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Geometric realizations of abstract regular polyhedra with automorphism group H_3

Jonn Angel L. Aranas and Mark L. Loyola*

Department of Mathematics, Ateneo de Manila University, Katipunan Avenue, Loyola Heights, Quezon City 1108, Philippines. *Correspondence e-mail: mloyola@ateneo.edu

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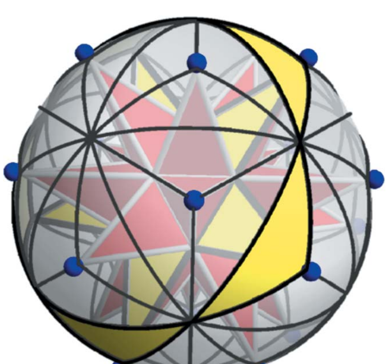


A geometric realization of an abstract polyhedron $\mathcal{P}$ is a mapping that sends an i-face to an open set of dimension i. This work adapts a method based on Wythoff construction to generate a full rank realization of an abstract regular polyhedron from its automorphism group Γ. The method entails finding a real orthogonal representation of Γ of degree 3 and applying its image to suitably chosen (not necessarily connected) open sets in space. To demonstrate the use of the method, it is applied to the abstract polyhedra whose automorphism groups are isomorphic to the non-crystallographic Coxeter group H_3.

1. Introduction

A geometric polyhedron is typically described as a three-dimensional solid of finite volume bounded by flat regions called its facets. Well known examples of geometric polyhedra include the five Platonic solids, which have been studied since antiquity, and their various truncations and stellations (Coxeter, 1973). Because of their mathematical and aesthetic appeal, geometric polyhedra are widely used as models in various fields of science and the arts (Senechal, 2013). In the field of crystallography, they have been used in studying the symmetry and structural formation of crystalline materials (Schulte, 2014; Delgado-Friedrichs & O'Keeffe, 2017), nanotubes (Cox & Hill, 2009, 2011) and even viruses (Salthouse *et al.*, 2015).

In classical geometry, a facet is a convex or star polygon bounded by line segments called edges and corner points called vertices. The arranged facets enclose an open set (not necessarily connected) in space called the polyhedron's cell. Shown in Fig. 1(a) is the Kepler–Poinsot star polyhedron called the small stellated dodecahedron and denoted by $\{\frac{5}{2}, 5\}$. The fraction $\frac{5}{2}$ in this symbol means that each of the polyhedron's facets is the union of five open triangular regions (dimension 2) whose topological closure is a regular pentagram [Fig. 1(b)]. The number 5 in the symbol gives the number of pentagrams that meet at each vertex of the polyhedron. Fig. 1(c) shows the five pentagrams which meet at the topmost vertex. Observe that these five pentagrams bound a pentagon-based pyramidal solid. The union of the open interiors (dimension 3) of 12 such pyramids makes up the cell of $\{\frac{5}{2}, 5\}$ [Fig. 1(d)].

Modern treatments of geometric polyhedra relax the classical conditions and allow facets that are surrounded by skew or non-coplanar edges or facets that self-intersect, have holes, or have no defined interiors (Grünbaum, 1994; Johnson, 2008). In fact, there is no universally agreed definition of a geometric polyhedron. The definition a work uses usually depends on the author's particular preferences, requirements and objectives.

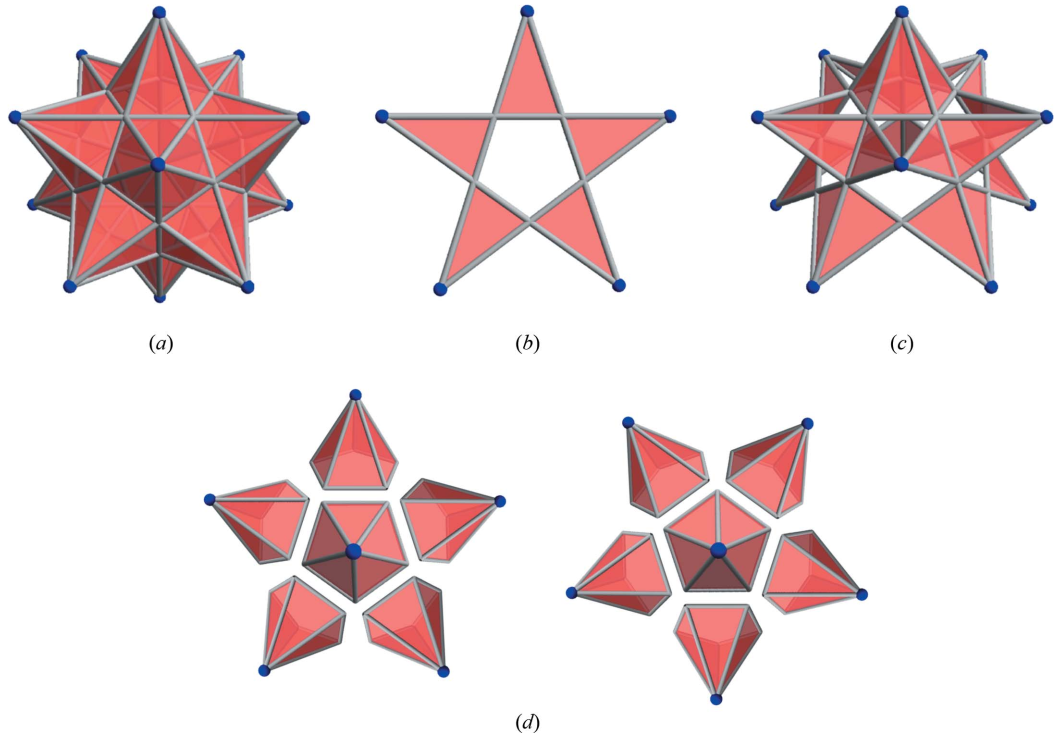


Figure 1
(a) Small stellated dodecahedron $\{\frac{5}{2}, 5\}$. (b) Pentagram $\{\frac{5}{2}\}$. (c) Five pentagrams meeting at the topmost vertex of (a). (d) Exploded view of (a), front and back, consisting of 12 pentagon-based pyramids.

While there is no consensus on what constitutes a geometric polyhedron, mathematicians generally agree on the conditions one must impose on its underlying vertex–edge–facet–cell incidence structure. This set of conditions defines a related mathematical object called an abstract polyhedron. Essentially, it is a partially ordered set of elements called faces that play analogous roles to the vertices, edges and facets of its geometric counterpart. Since an abstract polyhedron is combinatorial in nature, it is devoid of metric properties and, if regular (or at least highly symmetric), is best described by its group of automorphisms or incidence-preserving face mappings.

To lay out the foundation for a more rigorous treatment of geometric polyhedra, Johnson (2008) proposed the concept of a real polyhedron using an abstract polyhedron as blueprint. In his theory, a real polyhedron is the realization or the resulting figure when the faces of an abstract polyhedron are mapped to open sets in space. These associated open sets are selected so that they satisfy a set of conditions pertaining to their boundaries and intersections. Although Johnson's definition may not satisfy everyone's requirements, anchoring it to a well accepted concept makes it less ambiguous, and more consistent with existing notions and theories.

In this work, we shall adopt a simplified version of Johnson's real polyhedron for the definition of a geometric polyhedron. Our main objective is to adapt a method based on Wythoff construction (Coxeter, 1973) to generate a geometric polyhedron from a given abstract polyhedron $\mathcal{P}$ satisfying a regularity property. The adapted method builds the figure by applying the image of an orthogonal representation of the automorphism group of $\mathcal{P}$ to a collection of open sets in space. The method is formulated and stated in a way that is amenable to algorithmic computations and suited to computer-based graphics generation. This work extends and further illustrates the ideas found in the work of Clancy (2005) and concretizes the algebraic version of Wythoff construction found in Chapter 5A of McMullen & Schulte (2002).

To illustrate the use of the method, we apply it to the abstract regular polyhedra whose automorphism groups are isomorphic to the non-crystallographic Coxeter group H_3 (Humphreys, 1992). The group has order 120 and can be described via the group presentation

$$H_3 = \left\langle s_0, s_1, s_2 \,\middle|\, \begin{matrix} s_0^2 = s_1^2 = s_2^2 = e, \\ (s_0 s_1)^3 = (s_1 s_2)^5 = (s_0 s_2)^2 = e \end{matrix} \right\rangle. \quad (1)$$

Thus, H_3 is the automorphism group of the (abstract) regular icosahedron represented in the standard way. Since it is the group of symmetries of icosahedral structures, H_3 has played a fundamental role in the study of mathematical models of quasicrystals (Chen *et al.*, 1998; Patera & Twarock, 2002), carbon onions and carbon nanotubes (Twarock, 2002), and viruses (Janner, 2006; Keef & Twarock, 2009).

2. Abstract regular polyhedra and string C-groups

We begin with a non-empty finite set $\mathcal{P}$ of elements called faces that are partially ordered by a binary relation $\leq$. Two faces F and F' in $\mathcal{P}$ are said to be incident if either $F \leq F'$ or $F' \leq F$. The incidence relations among the faces can be graphically represented using a Hasse diagram in which a face

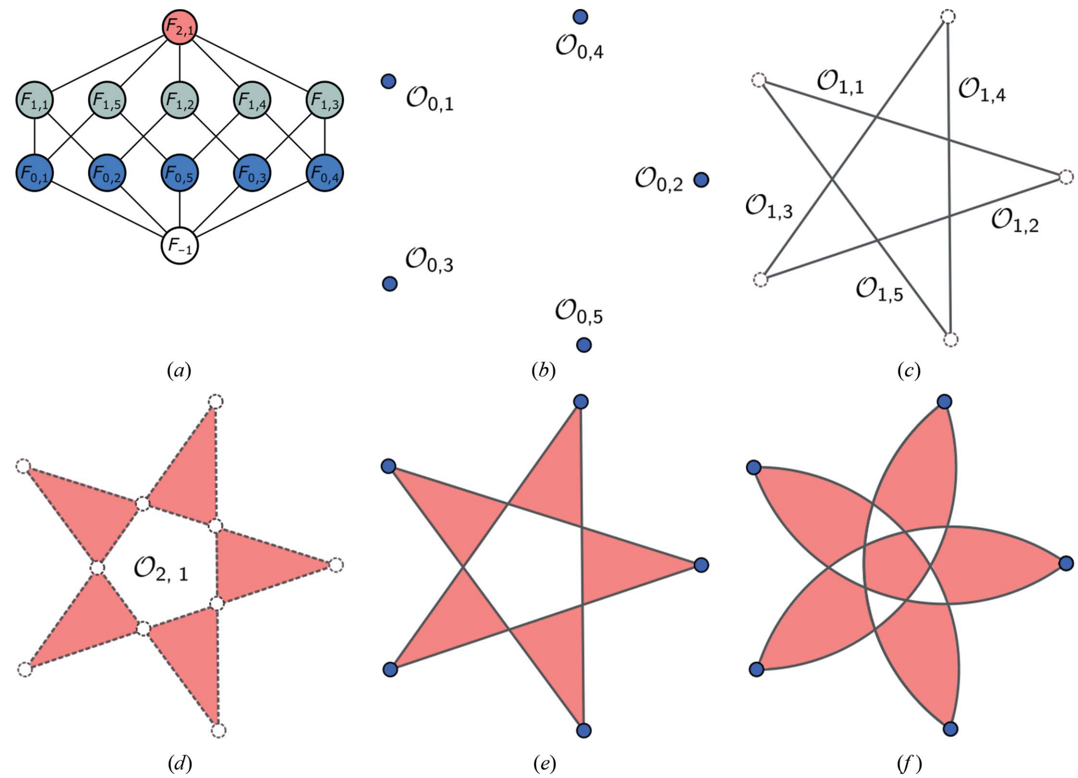


Figure 2
(*a*) Hasse diagram of a section of {5, 3}*120. Geometric (*b*) vertices, (*c*) edges, (*d*) facet corresponding to abstract faces that appear in the diagram. (*e*) Regular pentagram obtained by combining the geometric faces in (*a*)–(*d*). (*f*) The regular pentagram in (*e*) with straight edges replaced by circular arcs.

is represented by a node and two nodes on adjacent levels are connected by an edge if the corresponding faces are incident [Fig. 2(*a*)]. Since a partial order is transitive, we shall omit edges corresponding to implied incidences.

Given faces $F \leq F'$, we define the section F'/F of $\mathcal{P}$ to be the set of all faces H incident to both F and F', that is, $F'/F = \{H \in \mathcal{P} \mid F \leq H \leq F'\}$. Note that a section is also a partially ordered set under the same binary relation.

A flag is a maximal totally ordered subset of $\mathcal{P}$. Two flags are adjacent if they differ at exactly one face. Finally, $\mathcal{P}$ is said to be flag-connected if, for every pair of flags Φ, Ψ, there is a finite sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that successive flags are adjacent.

2.1. Abstract polyhedra

For our purposes, we shall now restrict our discussion to partially ordered non-empty finite sets $\mathcal{P}$ that satisfy the following three properties:

(P1) $\mathcal{P}$ contains a unique least face and a unique greatest face.

(P2) Each flag of $\mathcal{P}$ contains exactly five faces including the least face and the greatest face.

(P3) $\mathcal{P}$ is strongly flag-connected. That is, each section of $\mathcal{P}$ is flag-connected.

Properties P1 and P2 imply that any face F belongs to at least one flag and that the number of faces, excluding the least face, preceding it in any flag is constant. This constant, which we assign to be -1 for the least face, is called the rank of F. We shall call a face of rank i an i-face and denote it by F_i or $F_{i,j}$ (with index j for emphasis if there is more than one i-face). Thus, we denote the least face by F_{-1} and the greatest face by F_3. When drawing a Hasse diagram, we shall adopt the convention of putting faces of the same rank at the same level and faces of different ranks at different levels arranged in ascending order of rank.

A finite abstract polyhedron or a finite polytope of rank 3 (McMullen & Schulte, 2002) is a partially ordered finite set $\mathcal{P}$ that satisfies properties P1, P2 and P3 above, and property P4, also called the diamond property, below:

(P4) If $F_{i-1} \leq F_{i+1}$, where $0 \leq i \leq 2$, then there are precisely two i-faces F_i in $\mathcal{P}$ such that $F_{i-1} \leq F_i \leq F_{i+1}$.

This definition of an abstract polyhedron is, in fact, a specific case of the more general definition of an abstract n-polytope or polytope of rank n. By the rank of a polytope, we mean the rank of its greatest face. Borrowing terms from the theory of geometric polytopes, we shall refer to the -1-face of an abstract polyhedron as the empty face, a 0-face as a vertex, a 1-face as an edge, a 2-face as a facet and the 3-face as the cell (here, 'cell' is not meant to imply homeomorphism with a 3-ball).

2.2. String C-groups

We can endow an abstract polyhedron $\mathcal{P}$ with an algebraic structure by defining a map on its faces that preserves both ranks and incidence relations. A bijective map $\gamma : \mathcal{P} \to \mathcal{P}$ is

Table 1
The abstract regular H_3 polyhedra with automorphism group generated by $T = \{t_0, t_1, t_2\}$.

$\mathcal{P}$	t_0	t_1	t_2	$\dim \mathcal{W}(\varphi_1, (H_3, T))$	$\dim \mathcal{W}(\varphi_2, (H_3, T))$
{3, 5}*120	s_0	s_1	s_2	1	1
{3, 10}*120$_a$	s_0	s_1	s_0s_2	0	0
{3, 10}*120$_b$	s_0s_2	$(s_1s_2)^2s_0s_1s_2s_1$	s_0	0	0
{5, 3}*120	s_2	s_1	s_0	1	1
{5, 5}*120	s_0	$s_1s_2s_1$	s_2	1	1
{5, 6}*120$_b$	s_0	$s_1s_2s_1$	s_0s_2	0	0
{5, 6}*120$_c$	s_0s_2	$s_1s_0s_2s_1$	s_2	0	0
{5, 10}*120$_a$	s_0	$s_1s_2s_1$	$(s_1s_0s_2)^4s_1s_0$	0	0
{5, 10}*120$_b$	s_0s_2	$s_1s_0s_2s_1$	s_0	0	0
{6, 5}*120$_b$	s_0	$(s_1s_0s_2)^3s_1$	s_0s_2	0	0
{6, 5}*120$_c$	s_0s_2	$s_1s_2s_1$	s_0	1	1
{10, 3}*120$_b$	s_0s_2	s_1	s_0	1	1
{10, 3}*120$_c$	s_0	$s_1s_0s_2s_1$	$(s_1s_0s_2)^4s_1s_0$	0	0
{10, 5}*120$_a$	s_0	$s_1s_0s_2s_1$	s_0s_2	0	0
{10, 5}*120$_b$	s_0s_2	s_1	s_2	1	1

called an automorphism if it is incidence-preserving on the faces:

$$F \leq F' \text{ if and only if } \gamma(F) \leq \gamma(F'). \quad (2)$$

Using the properties of $\mathcal{P}$, it is easy to verify that an automorphism is necessarily rank-preserving as well. By convention, we shall use the right action notation $F\gamma$ for the image $\gamma(F)$. We shall denote the group of all automorphisms of $\mathcal{P}$ by $\Gamma(\mathcal{P})$, or just Γ when $\mathcal{P}$ is clear from the context.

An abstract polyhedron $\mathcal{P}$ is said to be regular if Γ acts transitively on its set of flags. Consequently, if $\mathcal{P}$ is regular, one can verify that the number p of vertices incident to a facet and the number q of facets incident to a vertex are both constant. These determine the (Schläfli) type $\{p, q\}$ of the regular polyhedron. Following the notation used in the *Atlas of Small Regular Polytopes* (Hartley, 2006), we denote by $\{p, q\}{*}m_x$ a regular polyhedron of type $\{p, q\}$ with automorphism group of order m. The index x, when present, distinguishes a polyhedron from other polyhedra of the same type with automorphism group of the same order.

For a regular polyhedron of type $\{p, q\}$, the automorphism group is a rank-3 string C-group of type $\{p, q\}$ and is best described as a pair (Γ, T), which consists of a group Γ and an ordered triple T of distinct generating involutions t_0, t_1, t_2 that satisfy three properties:

(i) String property: $t_0t_2 = t_2t_0$.

(ii) Intersection property: $\langle t_0, t_1\rangle \cap \langle t_1, t_2\rangle = \langle t_1\rangle$.

(iii) Order property: $\mathrm{ord}(t_0t_1) = p$, $\mathrm{ord}(t_1t_2) = q$.

Two string C-groups $(\Gamma, \{t_0, t_1, t_2\})$ and $(\Gamma', \{t_0', t_1', t_2'\})$ are considered equivalent if they have the same type and the map determined by $t_i \mapsto t_i'$ for $0 \leq i \leq 2$ is a group isomorphism. Since equivalence of string C-groups is dependent on the distinguished generating triples, we emphasize that two string C-groups may be considered distinct even if they are isomorphic as abstract groups.

A fundamental result in the theory of abstract polytopes is the bijective correspondence between regular polyhedra and rank-3 string C-groups. It follows that the enumeration of regular polyhedra is equivalent to the enumeration of rank-3 string C-groups. Thus, given an arbitrary group Γ, one may determine all regular polyhedra with automorphism group isomorphic to Γ by listing all generating triples T of distinct involutions t_0, t_1, t_2 that satisfy the string and intersection conditions. For groups of relatively small order, it is straightforward to implement a listing procedure to accomplish this task in the software *GAP* (The *GAP* Group, 2019). We apply this procedure to the non-crystallographic Coxeter group H_3 and obtain 15 abstract regular H_3 polyhedra, with each belonging to one of nine types, as summarized in Table 1.

2.3. Coset-based construction method

Given a string C-group (Γ, T), one may construct an abstract regular polyhedron $\mathcal{P}$ with automorphism group Γ. This is done by defining the cosets of certain subgroups of Γ as the faces of $\mathcal{P}$ and partially ordering these cosets using a suitably chosen binary relation. In the theorem below, we employ the construction method in Chapter 2E of McMullen & Schulte (2002).

Theorem 2.1. Suppose $(\Gamma, \{t_0, t_1, t_2\})$ is a string C-group of type $\{p, q\}$. Let $\Gamma_{-1} = \Gamma$, $\Gamma_3 = \Gamma$ and $\Gamma_i = \langle t_k \mid k \neq i\rangle$ for $0 \leq i \leq 2$. Then the following sequence of steps produces an abstract regular polyhedron $\mathcal{P}$ of type $\{p, q\}$ and automorphism group Γ:

(i) Generate a complete list of right coset representatives $\gamma_{i,j}$ of Γ_i indexed by $1 \leq j \leq [\Gamma : \Gamma_i]$ for $-1 \leq i \leq 3$.

(ii) Define $\mathcal{P}$ to be the set consisting of $F_{-1} = F_{-1,1} = \Gamma_{-1}$, $F_3 = F_{3,1} = \Gamma_3$ and $F_{i,j} = \Gamma_i\gamma_{i,j}$.

(iii) Define a binary relation $\leq$ on $\mathcal{P}$ where $F_{i,j} \leq F_{i',j'}$ if and only if $i \leq i'$ and $\Gamma_i\gamma_{i,j} \cap \Gamma_{i'}\gamma_{i',j'} \neq \emptyset$.

Moreover, the number of i-faces of $\mathcal{P}$ is equal to the index of Γ_i in Γ.

As a consequence of this theorem, we may identify a regular polyhedron $\mathcal{P}$ with Γ and an i-face $F_{i,j}$ with a coset representative $\gamma_{i,j}$ of Γ_i. For simplicity, we may assume this representative is the identity e when $j = 1$ and call $F_{i,1}$ the base i-face.

We further remark that if $\mathcal{P}'$ is a regular polyhedron whose automorphism group is described by the same string C-group (Γ, T), then the constructed polyhedron $\mathcal{P}$ in the theorem is actually isomorphic to $\mathcal{P}'$.

The Hasse diagram in Fig. 2(a) is a section of the H_3 polyhedron {5, 3}*120 in Table 1 consisting of a single empty face, 20 vertices, 30 edges, 12 facets and a single cell. This polyhedron, which is called the standard (abstract) regular dodecahedron, results from applying Theorem 2.1 to the string C-group $(H_3, \{s_2, s_1, s_0\})$.

3. Regular geometric polyhedra and Wythoff construction

Consider a finite abstract regular polyhedron $\mathcal{P}$ whose set of abstract i-faces is $\mathcal{P}_i$, where $-1 \le i \le 3$. Let Γ be its automorphism group with distinguished generating triple $T = \{t_0, t_1, t_2\}$. By an open set of dimension $0 \le i \le n$ in the Euclidean n-space $\mathbb{E}^n$, we mean a subset that is homeomorphic to a subset of $\mathbb{E}^i$ which is open in the usual sense. We denote by $\mathcal{O}(\mathbb{E}^n)$ the set of all such open subsets including the empty set $\emptyset$.

3.1. Regular geometric polyhedra

Define the map $\rho_{-1} : \mathcal{P}_{-1} \to \mathcal{O}(\mathbb{E}^n)$ that sends the empty face F_{-1} to the empty set $\mathcal{O}_{-1} = \emptyset$. Then for each $0 \le i \le 3$, recursively define a map $\rho_i : \mathcal{P}_i \to \mathcal{O}(\mathbb{E}^n)$ that sends each i-face $F_{i,j}$ with index $1 \le j \le [\Gamma : \Gamma_i]$ to a non-empty open set $\mathcal{O}_{i,j}$ of dimension i. We require that the boundary of $\mathcal{O}_{i,j}$ be $\cup_{0 \le k < i}(\cup_{F_{k,l} \le F_{i,j}} \mathcal{O}_{k,l})$, the union of the ρ_k images of the lower-rank k-faces $F_{k,l}$ incident to $F_{i,j}$.

Illustration 3.1. We illustrate the images of the i-faces of {5, 3}*120 that appear in the section represented by the Hasse diagram in Fig. 2(a). These images partially determine maps ρ_i for $0 \le i \le 2$.

Take the points $\mathcal{O}_{0,j}$, $1 \le j \le 5$, in $\mathbb{E}^3$ [Fig. 2(b)] and let ρ_0 send each vertex $F_{0,j}$ to $\mathcal{O}_{0,j}$, ρ_1 send each edge $F_{1,j}$ to the open line segment $\mathcal{O}_{1,j}$ in Fig. 2(c), and ρ_2 send the facet $F_{2,1}$ to the disconnected open set $\mathcal{O}_{2,1}$ in Fig. 2(d). We remark that $\mathcal{O}_{2,1}$ is simply the disjoint union of the five open triangular regions that make up the star-shaped decagon in Fig. 2(d).

When these open sets of different dimensions are combined, we obtain the pentagram shown in Fig. 2(e). Choosing open circular arcs as the images of the edges instead, and the disjoint union of suitably chosen open regions as the image of the lone facet, we obtain the figure illustrated in Fig. 2(f).

The mapping $\rho : \mathcal{P} \to \mathcal{O}(\mathbb{E}^n)$ whose restriction to $\mathcal{P}_i$ is ρ_i is called a geometric realization of $\mathcal{P}$. To simplify the discussion, we limit ourselves to when $n = 3$, in which case ρ is called a realization of full rank. To distinguish between an i-face in $\mathcal{P}$ and its image under ρ, we call the former an abstract i-face and the latter the realization of this abstract i-face, or a geometric i-face. Note that the rank of an abstract face corresponds to the dimension of a geometric face in a realization. We now refer to the union of the geometric faces, which we denote by $\rho(\mathcal{P})$, as a regular geometric polyhedron or, after identifying ρ with its image, a geometric realization of $\mathcal{P}$.

We remark that the definition of a realization stated above is an interpretation of the standard definition (Chapter 5A of McMullen & Schulte, 2002) in which abstract vertices are identified as points in space, edges as pairs of points, facets as sets of these pairs, and the cell as a collection of these sets of pairs. The standard definition, therefore, provides a blueprint to build a geometric polyhedron starting from its vertices and lets one exercise the freedom to choose Euclidean figures to represent abstract faces. Taking advantage of this freedom, we specify that abstract faces be associated to open sets with the appropriate dimension and boundary. This is to make the notion of a realization as wide-ranging as possible in order to cover typical figures representing known geometric polyhedra such as regular convex and star polyhedra. As we will see later, this will also allow one to generate polyhedra using curved edges and surfaces. Our definition of a realization is, in fact, consistent with the theory of real polytopes formulated by Johnson (2008). Essentially, Johnson defines a realization to be an assembly of open sets in space with imposed restrictions pertaining to their boundaries and intersections.

3.2. Wythoff construction

A faithful realization ρ is one where each induced map ρ_i is injective. That is, distinct abstract i-faces $F_{i,j}$ are sent to distinct geometric i-faces $\mathcal{O}_{i,j}$. It follows that there is a bijective correspondence between the set of $F_{i,j}$'s and the set of $\mathcal{O}_{i,j}$'s that preserves ranks and incidence relations in the former, and dimensions and boundary relations in the latter.

A symmetric realization, on the other hand, is one where each automorphism $\gamma \in \Gamma$ corresponds to an isometry of $\mathbb{E}^3$ that symmetrically permutes the $\mathcal{O}_{i,j}$'s. More specifically, since $\mathcal{P}$ is assumed to be finite, a symmetric realization presupposes the existence of an orthogonal representation $\varphi : \Gamma \to O(3)$ that satisfies

$$\rho_i(\mathrm{Im}(\gamma,\ F_{i,j})) = \mathrm{Im}(\varphi(\gamma),\ \rho_i(F_{i,j})) = \mathrm{Im}(\varphi(\gamma),\ \mathcal{O}_{i,j}), \qquad (3)$$

where $\mathrm{Im}(f, x)$ for a map f simply denotes the image $f(x)$.

We recall that $\varphi(\gamma)$ acts on $\mathbb{E}^3$ and preserves the usual Euclidean inner product. Consequently, for a fixed orthogonal basis, we may represent each γ with a 3×3 real orthogonal matrix. We denote the image $\varphi(\Gamma)$ of this representation by $G(\rho(\mathcal{P}))$, or just G when $\rho(\mathcal{P})$ is clear from the context. We remark that G is the symmetry group of the geometric polyhedron whenever ρ itself is faithful and symmetric. Such a realization always implies that φ is faithful:

Proposition 3.1. Let ρ be a faithful symmetric realization of $\mathcal{P}$. If $\varphi : \Gamma \to O(3)$ is the associated orthogonal representation, then φ is faithful.

Proof. It suffices to show that if $\varphi(\gamma)$ is the identity isometry ι, then γ is the identity automorphism e. By equation (3), we have

$$\rho_i(\mathrm{Im}(\gamma,\ F_{i,j})) = \mathrm{Im}(\varphi(\gamma),\ \rho_i(F_{i,j})) = \mathrm{Im}(\iota,\ \rho_i(F_{i,j})) = \rho_i(F_{i,j}), \tag{4}$$

for any abstract i-face $F_{i,j}$. Thus, $\rho_i(\mathrm{Im}(\gamma,\ F_{i,j})) = \rho_i(F_{i,j})$, which is equivalent to $\mathrm{Im}(\gamma,\ F_{i,j}) = F_{i,j}$ by faithfulness of ρ. Since $F_{i,j}$ is arbitrary, γ must be e. Consequently, φ is faithful. □

From this point forward, we restrict ourselves to realizations ρ which are both faithful and symmetric. With these properties not only do we have a correspondence between abstract and geometric faces, we also have a correspondence between the action of the automorphism group on the abstract faces and the action of the symmetry group on the corresponding geometric faces. Consequently, any geometric polyhedron obtained from ρ will automatically satisfy regularity or transitivity of geometric flags. Thus, to construct ρ, we must employ a faithful orthogonal representation by Proposition 3.1. The group H_3 has two such irreducible representations (Koca & Koca, 1998),

$$\varphi_1\ :\ s_0 \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad s_1 \mapsto \frac{1}{2}\begin{bmatrix} 1 & -\tau & -\sigma \\ -\tau & \sigma & 1 \\ -\sigma & 1 & \tau \end{bmatrix},\quad s_2 \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

$$\varphi_2\ :\ s_0 \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad s_1 \mapsto \frac{1}{2}\begin{bmatrix} 1 & -\sigma & -\tau \\ -\sigma & \tau & 1 \\ -\tau & 1 & \sigma \end{bmatrix},\quad s_2 \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6}$$

where $\tau = (1 + 5^{1/2})/2$ and $\sigma = (1 - 5^{1/2})/2$.

We now describe an explicit construction method in Theorem 3.1 to obtain a realization of a polyhedron from a string C-group (Γ, T). Recall earlier that we may identify an i-face $F_{i,j}$ with a coset representative $\gamma_{i,j}$ of Γ_i.

Theorem 3.1. Let $(\Gamma,\ T)$ be a string C-group which characterizes the automorphism group of an abstract regular polyhedron $\mathcal{P}$ and let φ be a faithful irreducible orthogonal representation of Γ. Then the following sequence of steps produces a faithful symmetric realization ρ of $\mathcal{P}$:

(i) Generate a complete list of right coset representatives $\gamma_{i,j}$ of Γ_i with index $1 \le j \le [\Gamma : \Gamma_i]$ for $0 \le i \le 3$.

(ii) Compute the matrix representations $\varphi(\gamma_{i,j})$ of the coset representatives $\gamma_{i,j}$.

(iii) Compute the Wythoff space

$$\mathcal{W}(\varphi,\ (\Gamma,\ T)) = \{\mathbf{x} \in \mathbb{E}^3 \mid \mathrm{Im}(\varphi(t_1), \mathbf{x}) = \mathrm{Im}(\varphi(t_2), \mathbf{x}) = \mathbf{x}\} \tag{7}$$

associated with the pair $(\varphi,\ (\Gamma,\ T))$. This space consists of points in $\mathbb{E}^3$ that are fixed by both $\varphi(t_1)$ and $\varphi(t_2)$.

(iv) Suppose $\dim \mathcal{W}(\varphi,\ (\Gamma,\ T)) > 0$. Pick a point $\mathbf{x} \in \mathcal{W}(\varphi,\ (\Gamma,\ T))$ and let $\mathcal{O}_{0,1}$ be $\mathbf{x}$.

(v) For $1 \le i \le 3$:

(*a*) Determine the indexing set $J_i = \{j \mid \Gamma_{i-1}\gamma_{i-1,j} \cap \Gamma_i \ne \emptyset\}$.

(*b*) Compute the open sets $\mathcal{O}_{i-1,j} = \mathrm{Im}(\varphi(\gamma_{i-1,j}), \mathcal{O}_{i-1,1})$ for each $j \in J_i$.

(*c*) Let $\mathcal{O}_{i,1}$ be an open set of dimension i that is bounded by $\mathcal{O}_{i-1,j}$ for $j \in J_i$ and has $G_i = \varphi(\Gamma_i)$ as its stabilizer in G.

(vi) For $0 \le i \le 3$, define ρ_i to be the map $\mathcal{P}_i \to \mathcal{O}(\mathbb{E}^n)$ that sends each $F_{i,j}$ to $\mathcal{O}_{i,j} = \mathrm{Im}(\varphi(\gamma_{i,j}), \mathcal{O}_{i,1})$ for $1 \le j \le [\Gamma : \Gamma_i]$.

(vii) Define ρ to be the map $\mathcal{P} \to \mathcal{O}(\mathbb{E}^n)$ whose restriction to $\mathcal{P}_i$ is ρ_i.

Proof. To prove the theorem, we need only show that ρ is faithful and symmetric. To this end, let $\gamma \in \Gamma$ and $F_{i,j}, F_{i,k} \in \mathcal{P}_i$.

Suppose that $\rho_i(F_{i,j}) = \rho_i(F_{i,k})$. That is, two abstract i-faces are sent to the same open set of dimension i. By the definition of $\mathcal{O}_{i,j}$ in step (vi), we obtain

$$\rho_i(F_{i,j}) = \mathrm{Im}(\varphi(\gamma_{i,j}), \mathcal{O}_{i,1}) \quad \text{and} \quad \rho_i(F_{i,k}) = \mathrm{Im}(\varphi(\gamma_{i,k}), \mathcal{O}_{i,1}), \tag{8}$$

which implies that $\varphi(\gamma_{i,j}\gamma_{i,k}^{-1})$ stabilizes $\mathcal{O}_{i,1}$. Since $\mathcal{O}_{i,1}$ is chosen so that it has G_i as its stabilizer in G, we must have $\gamma_{i,j}\gamma_{i,k}^{-1} \in \Gamma_i$. Thus, $\Gamma_i\gamma_{i,j} = \Gamma_i\gamma_{i,k}$ or, equivalently, $F_{i,j} = F_{i,k}$. Hence, ρ is faithful.

To show that ρ is symmetric as well, let $\mathrm{Im}(\gamma,\ F_{i,j}) = F_{i,k}$. It follows that $(\Gamma_i\gamma_{i,j})\gamma = \Gamma_i\gamma_{i,k}$ and so $\gamma = \gamma_{i,j}^{-1}\sigma\gamma_{i,k}$ for some $\sigma \in \Gamma_i$. The image of $\mathcal{O}_{i,j}$ under $\varphi(\gamma)$ is

$$\begin{aligned} \mathrm{Im}(\varphi(\gamma), \mathcal{O}_{i,j}) &= \mathrm{Im}(\varphi(\gamma_{i,j}^{-1}\sigma\gamma_{i,k}), \mathcal{O}_{i,j}) \\ &= \mathrm{Im}(\varphi(\gamma_{i,j}^{-1})\,\varphi(\sigma)\,\varphi(\gamma_{i,k}), \mathcal{O}_{i,j}) \\ &= \mathrm{Im}(\varphi(\sigma)\,\varphi(\gamma_{i,k}), \mathcal{O}_{i,1}) \\ &= \mathrm{Im}(\varphi(\gamma_{i,k}), \mathcal{O}_{i,1}), \end{aligned} \tag{9}$$

where each component of $\varphi(\gamma_{i,j}^{-1})\varphi(\sigma)\,\varphi(\gamma_{i,k})$ is sequentially applied to $\mathcal{O}_{i,j}$ from left to right to conform with the right action of Γ on $\mathcal{P}_i$. We then have

$$\begin{aligned} \rho_i(\mathrm{Im}(\gamma,\ F_{i,j})) = \rho_i(F_{i,k}) = \mathcal{O}_{i,k} &= \mathrm{Im}(\varphi(\gamma_{i,k}), \mathcal{O}_{i,1}) \\ &= \mathrm{Im}(\varphi(\gamma), \mathcal{O}_{i,j}). \end{aligned} \tag{10}$$

Hence, ρ is symmetric. □

Based on the above proof, it is important to remark that the imposition that the base geometric i-face $\mathcal{O}_{i,1}$ be chosen so that its stabilizer in G is G_i guarantees that the resulting realization ρ will still be faithful, even if the base abstract i-face $F_{i,1}$ is not uniquely determined by the abstract $(i-1)$-faces incident to it. In particular, it is possible to construct a faithful symmetric realization of an abstract regular polyhedron such as a hosohedron $\{2, q\}*4q$, which has q abstract edges incident to its two abstract vertices, or a dihedron $\{p, 2\}*4p$, which has two abstract facets incident to its p abstract edges. For instance, to obtain a faithful symmetric realization of $\{2, q\}*4q$, one may use an open spherical great semicircle (whose stabilizer is isomorphic to the dihedral group $\mathbf{D}_2$) but not an open line segment (whose stabilizer is the whole group G) as the image of an abstract edge.

The procedure described in Theorem 3.1 is an algebraic version of the method of Wythoff construction (Chapter 5A of McMullen & Schulte, 2002) named after the Dutch mathematician Willem Abraham Wythoff. His original geometric version is used to construct uniform tessellations. It relies on a kaleidoscope-like setup in which three reflection mirrors bound what becomes a fundamental triangle of the resulting tessellation (Coxeter, 1973). In Theorem 3.1, the fixed spaces of the generators in T, which may not necessarily be reflections, play the role of the mirrors.

For a string C-group of type $\{p, q\}$, we may compute the dimension of the Wythoff space using the formula (Clancy, 2005)

$$\dim \mathcal{W}(\varphi,\ (\Gamma,\ T)) = \frac{1}{2q} \sum_{\gamma \in \langle t_1, t_2 \rangle} \mathrm{Tr}\varphi(\gamma), \tag{11}$$

where $\mathrm{Tr}\varphi(\gamma)$ denotes the trace of $\varphi(\gamma)$. We note that if $\dim \mathcal{W}(\varphi,\ (\Gamma,\ T)) = 0$, we do not obtain any realization via Theorem 3.1. If $\dim \mathcal{W}(\varphi,\ (\Gamma,\ T)) = 1$, on the other hand, any two choices for the base geometric vertex will just be scalar multiples of each other. It follows that a one-dimensional Wythoff space produces only algebraically equivalent realizations. Different choices for the open image of a face, however, may yield polyhedra that are topologically different.

Illustration 3.2. We now illustrate the use of Theorem 3.1 to create a realization $\rho_{\rm st}$ of the standard (abstract) regular dodecahedron $\{5, 3\}*120$ with automorphism group $\Gamma = H_3$ generated by the triple T consisting of $t_0 = s_2$, $t_1 = s_1$, $t_2 = s_0$. Employing the representation φ_2, we have the following generating matrices for $G = \varphi_2(\Gamma)$:

$$\varphi_2(t_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\varphi_2(t_1) = \frac{1}{2}\begin{bmatrix} 1 & -\sigma & -\tau \\ -\sigma & \tau & 1 \\ -\tau & 1 & \sigma \end{bmatrix}, \tag{12}$$
$$\varphi_2(t_2) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These three generators correspond to reflections of $\mathbb{E}^3$, with the first and third having the xz plane and yz plane, respectively, as mirrors.

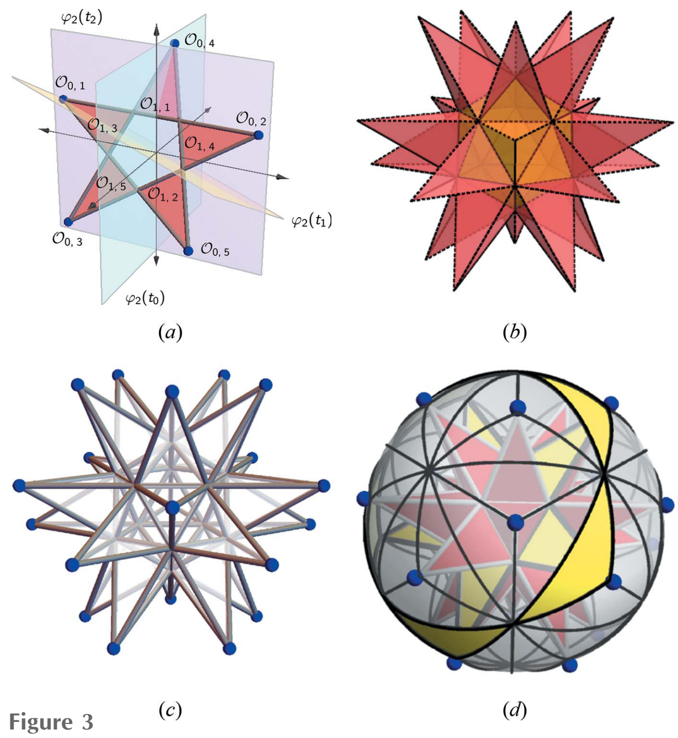


Figure 3
(*a*) The base geometric vertex, edge and facet of $\rho_{\rm st}(\{5, 3\}*120)$. (*b*) The base geometric cell of $\rho_{\rm st}(\{5, 3\}*120)$ with icosahedral hole. (*c*) Union of the geometric vertices and edges of $\rho_{\rm st}(\{5, 3\}*120)$. (*d*) Spherical realization $\rho_{\rm sp}(\{5, 3\}*120)$ circumscribing the star realization $\rho_{\rm st}(\{5, 3\}*120)$.

For each $0 \le i \le 3$, we use *GAP* to generate a complete list of right coset representatives $\gamma_{i,j}$ of Γ_i, where $1 \le j \le [\Gamma : \Gamma_i]$, and their corresponding matrix representations $\varphi_2(\gamma_{i,j})$.

By formula (11), we obtain $\dim \mathcal{W}(\varphi,\ (\Gamma,\ T)) = 1$. We compute the Wythoff space by finding a basis for the intersection of the 1-eigenspaces of $\varphi_2(t_1)$ and $\varphi_2(t_2)$. Using the Zassenhaus algorithm to compute for a basis for this intersection yields $\mathcal{W}(\varphi_2,\ (\Gamma,\ T)) = \mathrm{span}\{(0, 1, 1+\sigma)\} \subseteq \mathbb{E}^3$.

As explained earlier, we construct the base geometric i-face $\mathcal{O}_{i,1}$ for $1 \le i \le 3$ taking into account not only $\mathcal{O}_{i-1,1}$ but also the realizations $\mathcal{O}_{i-1,j}$ of the $(i-1)$-faces $\gamma_{i-1,j}$ incident to $\gamma_{i,1}$. This ensures that, at each stage, $\mathcal{O}_{i,1}$ is bounded by these $\mathcal{O}_{i-1,j}$'s as required by the definition of a realization. In addition, $\mathcal{O}_{i,1}$ must be chosen carefully so that its stabilizer is G_i.

(i) Base geometric vertex: pick the point $(0, 1, 1+\sigma)$ in the Wythoff space and let this be $\mathcal{O}_{0,1}$.

(ii) Base geometric edge: aside from $\gamma_{0,1} = e$, only the vertex $\gamma_{0,2} = t_0$ is incident to the base edge $\gamma_{1,1} = e$. We define $\mathcal{O}_{1,1}$ to be the open line segment [Fig. 3(*a*)] whose endpoints are $\mathcal{O}_{0,1}$ and $\mathcal{O}_{0,2} = \mathrm{Im}(\varphi_2(t_0),\ \mathcal{O}_{0,1}) = (0, -1, 1+\sigma)$. This segment is stabilized by G_1, with $\varphi_2(t_0)$ interchanging these endpoints and $\varphi_2(t_2)$ fixing them.

(iii) Base geometric facet: there are five edges incident to the base facet $\gamma_{2,1} = e$. These are $\gamma_{1,1} = e$, $\gamma_{1,2} = t_0t_1$, $\gamma_{1,3} = (t_0t_1)^2$, $\gamma_{1,4} = t_1t_0t_1$ and $\gamma_{1,5} = t_0t_1$. We define $\mathcal{O}_{2,1}$ to be the open regular pentagram [Fig. 3(*a*)] bounded by the segments $\mathcal{O}_{1,j} = \mathrm{Im}(\varphi_2(\gamma_{1,j}),\ \mathcal{O}_{1,1})$ for $1 \le j \le 5$ with endpoints $\mathcal{O}_{0,1}$, $\mathcal{O}_{0,2}$, $\mathcal{O}_{0,3} = (\sigma, -\sigma, \sigma)$, $\mathcal{O}_{0,4} = (1+\sigma, 0, 1)$, $\mathcal{O}_{0,5} = (\sigma, \sigma, \sigma)$ as shown in the figure. It is straightforward to verify that this pentagram is stabilized by G_2 with $\varphi_2(t_0)$ fixing $\mathcal{O}_{1,1}$, $\varphi_2(t_1)$ fixing $\mathcal{O}_{1,3}$ and either symmetry permuting the remaining segments.

(iv) Base geometric cell: there are 12 facets incident to the base cell $\gamma_{3,1} = e$. These are $\gamma_{2,1} = e$, $\gamma_{2,2} = t_2$, $\gamma_{2,3} = t_1t_2$, $\gamma_{2,4} = t_0t_1t_2$, $\gamma_{2,5} = t_1t_0t_1t_2$, $\gamma_{2,6} = t_2t_1t_0t_1t_2$, $\gamma_{2,7} = (t_0t_1)^2t_2$, $\gamma_{2,8} = t_0t_2t_1t_0t_1t_2$, $\gamma_{2,9} = t_1t_0t_2t_1t_0t_1t_2$, $\gamma_{2,10} = t_0t_1t_0t_2t_1t_0t_1t_2$, $\gamma_{2,11} = (t_1t_0)^2t_2t_1t_0t_1t_2$ and $\gamma_{2,12} = t_2(t_1t_0)^2t_2t_1t_0t_1t_2$. We define $\mathcal{O}_3$ to be the open set [Fig. 3(*b*)] bounded by the open pentagrams $\mathcal{O}_{2,j} = \mathrm{Im}(\varphi_2(\gamma_{2,j}), \mathcal{O}_{2,1})$ for $1 \le j \le 12$. The set $\mathcal{O}_3$ is the disjoint union of 20 open triangular pyramids whose bases form the bounding surface of a regular icosahedron. We can thus informally describe $\mathcal{O}_3$ as an open 'spiky' solid with an icosahedral hole at its core. It will follow that $\mathcal{O}_3$ is stabilized by G after verifying that each generator of G either fixes a bounding pentagram or sends it to another one.

The resulting geometric polyhedron $\rho_{\rm st}(\{5, 3\}^*120)$ is obtained by getting the union of the geometric vertices and edges [Fig. 3(*c*)] and the geometric facets and cell [Fig. 3(*b*)]. We call this geometric realization the great stellated dodecahedron, one of the Kepler–Poinsot geometric polyhedra, and denote it by $\{\frac{5}{2}, 3\}$.

3.3. Geometric faces

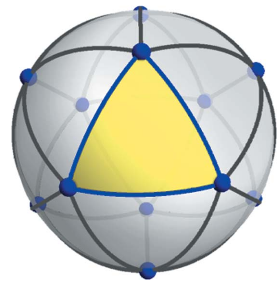
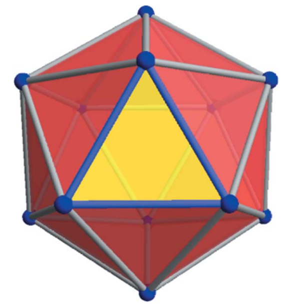
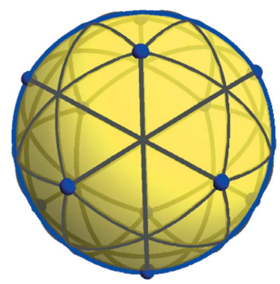
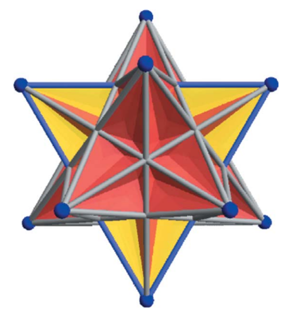
Here we describe four different families of realizations – spherical, convex, star and skew – classified according to the geometry and relative arrangements of their associated open sets. These were chosen to demonstrate the capability of Theorem 3.1 to later produce a realization for each of the regular H_3 polyhedra in Table 1. It is important to note that other families of open sets may also be chosen and the four enumerated here are by no means the only options available.

3.3.1. Spherical realization. Since orthogonal matrices are isometric, a sphere is a natural space for a geometric polyhedron to inhabit. For a spherical realization denoted by $\rho_{\rm sp}$, we define the base geometric vertex as a point on the surface of a fixed sphere, the base geometric edge as an open great circular arc, the base geometric facet as an open spherical polygon and the geometric cell as the sphere's interior. Observe that the geometric faces excluding the cell tile the surface of the sphere. Thus, we may regard a spherical realization as a covering of the surface of a sphere by spherical polygons.

3.3.2. Convex and star realizations. Suppose that, in a spherical realization, we set the base geometric edge to be an open line segment instead of a spherical arc. Provided that the resulting bounding edges of the base geometric facet are coplanar, we may define a classical realization which is either convex and denoted by $\rho_{\rm co}$ or star and denoted by $\rho_{\rm st}$. If a pair of edges (or facets) intersect, we set the base geometric facet (cell) to be the union of disconnected open regions bounded by its incident edges (facets). Otherwise, we define it as the interior of the convex hull of these edges (facets).

The presence of intersecting edges or facets characterizes a star realization. That is, the resulting star polyhedron is polymorphic and has a cell which generally consists of the union of two or more distinct open regions in space (Johnson, 2008).

Table 2
Full rank geometric realizations of $\{3, 5\}^*120$ ($v = 12$, $e = 30$, $f = 20$) with the base facet and its boundary highlighted.

φ_i	$\rho_{\rm sp}$	$\rho_{\rm co}/\rho_{\rm st}$
φ_1	 (*a*) Spherical icosahedron	 (*b*) Convex icosahedron
φ_2	 (*c*) Spherical great icosahedron	 (*d*) Star great icosahedron

The convexity of the resulting geometric polyhedron, on the other hand, characterizes a convex realization. That is, a convex polyhedron is a solid where each geometric i-face is the interior of the convex hull of its bounding geometric $(i-1)$-faces.

3.3.3. Skew realization. Consider the scenario in which the geometric edges are open line segments as in a convex or a star realization, but the resulting bounding edges of the base geometric facet are non-coplanar. In this case, we set the base geometric facet to be the interior of the minimal surface (local area-minimizing surface) obtained by solving Plateau's problem on the facet's bounding edges (Hass, 1991). A physical model of this minimal surface is the soap film obtained by dipping a wire frame bent into the shape of the base facet's boundary into a soap solution. This gives rise to what we now refer to as a skew realization $\rho_{\rm sk}$. Such a realization results in a polyhedron with facets that are curved as opposed to planar.

4. Regular geometric H_3 polyhedra

The method discussed in Theorem 3.1 allows one to reproduce the spherical and classical realizations of the abstract regular H_3 polyhedra and lets one construct non-standard realizations.

Applying formula (11) to the string C-groups in Table 1 yields six abstract polyhedra with non-zero Wythoff dimen-

Table 3
Full rank geometric realizations of $\{5, 3\}*120$ ($v = 20$, $e = 30$, $f = 12$) with the base facet and its boundary highlighted.

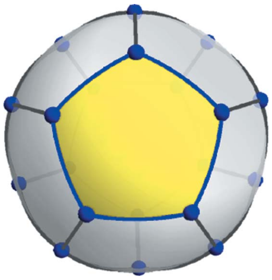
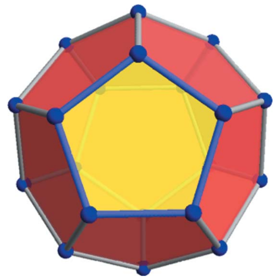
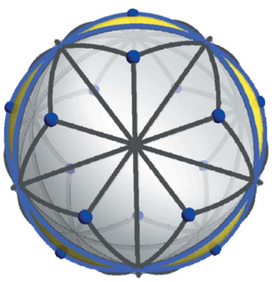
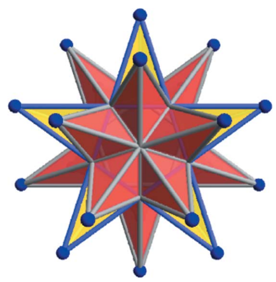
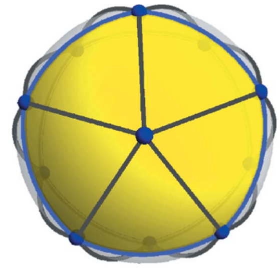
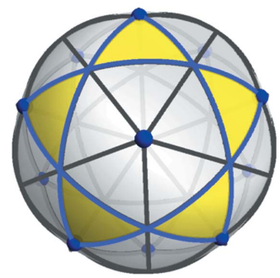
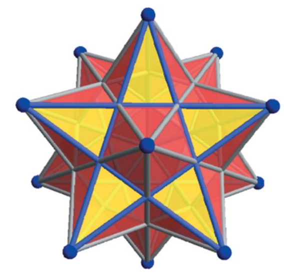
φ_i	ρ_{sp}	ρ_{co}/ρ_{st}
φ_1	 (*a*) Spherical dodecahedron	 (*b*) Convex dodecahedron
φ_2	 (*c*) Spherical great stellated dodecahedron	 (*d*) Star great stellated dodecahedron

Table 4
Full rank geometric realizations of $\{5, 5\}*120$ ($v = 12$, $e = 30$, $f = 12$) with the base facet and its boundary highlighted.

φ_i	ρ_{sp}	ρ_{co}/ρ_{st}
φ_1	 (*a*) Spherical great dodecahedron	 (*b*) Star great dodecahedron
φ_2	 (*c*) Spherical small stellated dodecahedron	 (*d*) Star small stellated dodecahedron

sion: $\{3, 5\}*120$, $\{5, 3\}*120$, $\{5, 5\}*120$, $\{6, 5\}*120_c$, $\{10, 3\}*120_b$ and $\{10, 5\}*120_b$. These realizable polyhedra have a one-dimensional Wythoff space for either representation φ_1, φ_2 and, consequently, will give rise to 12 spherical and 12 non-spherical (convex, star or skew) realizations. The resulting geometric polyhedra are rendered as solid figures using *Mathematica* (Wolfram Research, 2018) and presented in Tables 2–4 and 6–8. The number of vertices v, edges e and facets f of these polyhedra are also indicated in the tables.

The spherical realizations correspond to covers of the unit sphere by spherical projections of planar triangles, pentagons, pentagrams, skew hexagons and skew decagons. Some of these projected polygons cover the sphere only once [see Tables 2(*a*), 3(*a*), 3(*c*) and 4(*c*)] and hence generate a regular spherical tessellation.

The classical realizations consist of two convex polyhedra: the icosahedron [Table 2(*b*)] and the dodecahedron [Table 3(*b*)] with a triangle and a pentagon, respectively, as a facet; and the four Kepler–Poinsot star polyhedra: the great icosahedron $\{3, \frac{5}{2}\}$ [Table 2(*d*)], the great stellated dodecahedron $\{\frac{5}{2}, 3\}$ [Table 3(*d*)], the great dodecahedron $\{5, \frac{5}{2}\}$ [Table 4(*b*)] and the small stellated dodecahedron $\{\frac{5}{2}, 5\}$ [Table 4(*d*)], with a triangle, a pentagram, a pentagon and a pentagram, respectively, as a facet. These star polyhedra are also referred to as the stellations of the convex icosahedron and dodecahedron and may be constructed alternatively by extending the facets of the latter until they intersect and form the facets of the former.

To illustrate the similarities and differences between a spherical and a classical realization, we take the polyhedron $\{5, 3\}*120$ and embed its realization under ρ_{st} in Illustration 3.2

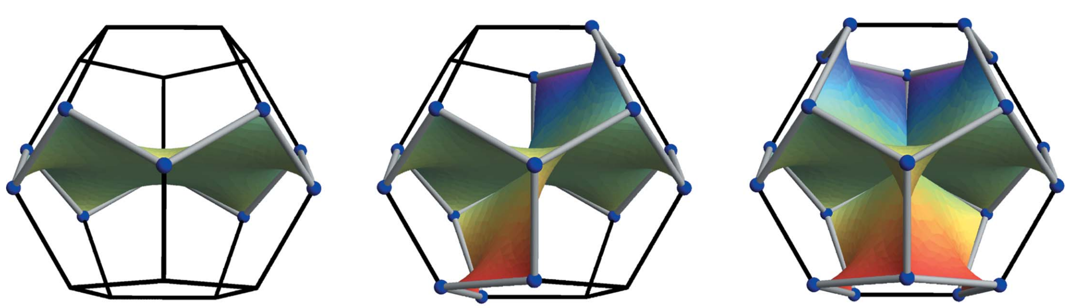
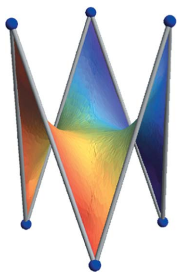
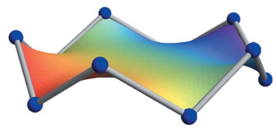
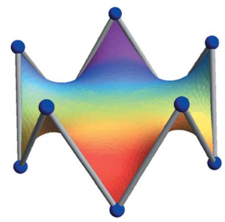
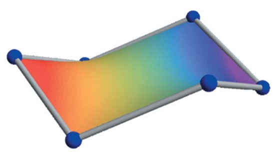
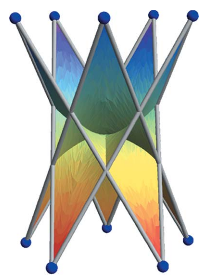
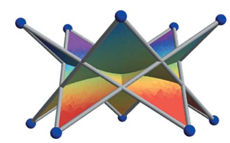


Figure 4
(*a*) The base geometric facet of $\rho_{sk}(\{10, 3\}*120_b)$ with (*b*) one and (*c*) two of its symmetric copies, bounding a set in space with non-zero volume. For ease of visualization, the facets are embedded inside a skeletal dodecahedron which shares the same geometric vertices as this realization.

Table 5
Geometric base facets of the skew realizations of $\{6, 5\}*120_c$, $\{10, 3\}*120_b$ and $\{10, 5\}*120_b$.

φ_i	$\{6, 5\}*120_c$	$\{10, 3\}*120_b$	$\{10, 5\}*120_b$
φ_1			
φ_2			

inside its realization under $\rho_{\rm sp}$. We present the embedded figures in Fig. 3(*d*). We also highlight the planar pentagram facet in the star polyhedron and its projection on the unit sphere in the spherical polyhedron. Note how the edges in both polyhedra intersect at points which do not correspond to vertices.

None of $\{6, 5\}*120_c$, $\{10, 3\}*120_b$ or $\{10, 5\}*120_b$ admit a convex or a star realization since their base geometric facets have non-coplanar bounding edges. By implementing a simple numerical iterative algorithm based on the finite-element method, we obtain a minimal surface as the base facet of each of these polyhedra. For instance, when this algorithm is applied to $\{10, 3\}*120_b$, we obtain the skew decagon facet in Fig. 4 and the geometric polyhedron in Table 7(*b*). The other five geometric realizations are displayed in Tables 6, 7 and 8. Their facets, which are either skew hexagons or skew decagons, are shown in Table 5.

It is worth mentioning that 12 of the geometric H_3 polyhedra enumerated above can be related to one another by three mixing operations: duality, Petrie operation and facetting. Each of these operations can be applied to a string C-group to obtain a new group that corresponds to a new geometric polyhedron. Applying a finite sequence of such operations to, say, the icosahedron yields the dodecahedron, the four Kepler–Poinsot polyhedra and their six Petrials. The reader is referred to Chapter 7E of McMullen & Schulte (2002) for a detailed discussion of these operations.

5. Conclusions and future outlook

This study has demonstrated a method of producing a full rank geometric realization of an abstract regular polyhedron. Existing work on realizations emphasizes their algebraic

Table 6
Full rank geometric realizations of $\{6, 5\}*120_c$ ($v = 12$, $e = 30$, $f = 10$) with the base facet and its boundary highlighted.

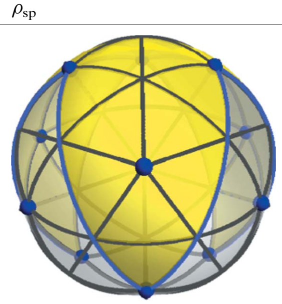
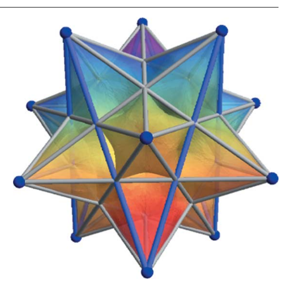
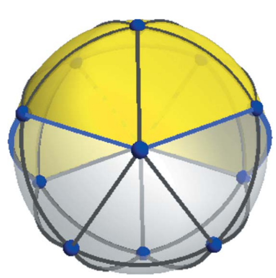
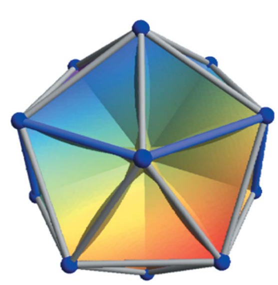
φ_i	$\rho_{\rm sp}$	$\rho_{\rm sk}$
φ_1		
φ_2		

Table 7
Full rank geometric realizations of $\{10, 3\}*120_b$ ($v = 20$, $e = 30$, $f = 6$) with the base facet and its boundary highlighted.

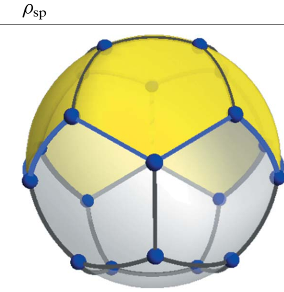
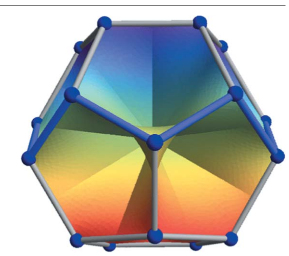
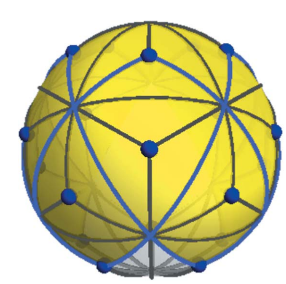
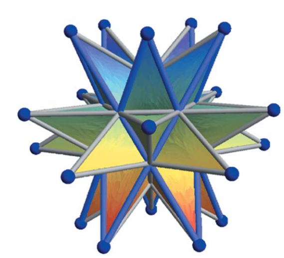
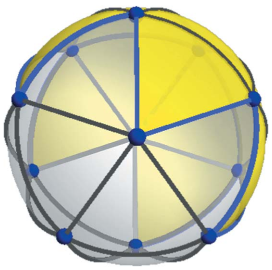
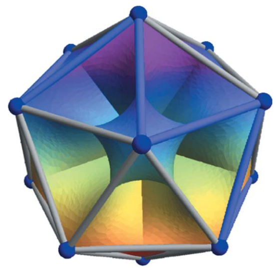
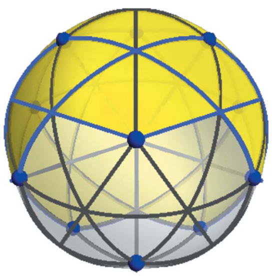
φ_i	$\rho_{\rm sp}$	$\rho_{\rm sk}$
φ_1		
φ_2		

Table 8
Full rank geometric realizations of {10, 5}*120$_b$ (v = 12, e = 30, f = 6) with the base facet and its boundary highlighted.

φ_i	$\rho_{\rm sp}$	$\rho_{\rm sk}$
φ_1		
φ_2		

aspects. For instance, the articles by McMullen (1989, 2011, 2014), McMullen & Monson (2003) and Ladisch (2016) focus on the structure of the realization cone or the set of all realizations of a given polytope up to congruence. In contrast, we have highlighted their geometric aspects by identifying the realizations with their images as solid figures in space.

Adapted from Wythoff construction, the method presented in this study is algorithmic in nature and hence well suited for computer implementation. This was exhibited when we applied the method to abstract regular polyhedra with automorphism group isomorphic to H_3. The entire process involved enumerating the abstract regular H_3 polyhedra through a search algorithm in *GAP* and using the irreducible orthogonal representations of H_3 to generate the corresponding figures of their geometric realizations in *Mathematica*. This allowed us to reproduce the classical convex and star polyhedra with icosahedral symmetry, as well as non-standard icosahedral polyhedra with minimal surfaces as facets. We reiterate that we do not limit ourselves to the families of open sets listed in Section 3.3 when considering a realization.

We remark that even though we have applied the method only to the abstract regular H_3 polyhedra, the method is also applicable to other regular polyhedra and may be extended to polytopes of higher rank. In particular, one may apply the method to regular polyhedra arising from the groups A_n, B_n and H_n (Humphreys, 1992). For future work, it is worthwhile considering establishing an analogous construction method for the realizations of abstract semiregular polytopes using a version of Wythoff construction found in Monson & Schulte (2012) as a framework.

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